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On the effect of particle couples on the motion of a dilute suspension of spheroids

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This paper is concerned with the effect on the bulk motion of applying torque by external means to the particles of a suspension. Our investigation is based upon theoretical consideration of the motion of a dilute suspension of permanently magnetized spheroids, in the presence of a uniform magnetic field, for cases in which the motion of a Newtonian fluid would be uni-directional. For convenience, we have divided the work into two parts: in the first, the particles are assumed to be perfectly spherical and the magnetic field strength to be arbitrary; in the second, the particles are taken as arbitrary spheroids (with their magnetic dipole axis coincident with the axis of revolution), but the magnetic field is assumed to be strong enough to ensure that the magnetic dipole axis of each particle is effectively aligned with the field vector **H**. We concentrate attention on the development of general criteria which allow an *a priori* determination as to whether the bulk motion remains uni-directional in the presence of particle couples, and (when the motion is uni-directional) whether the resultant velocity profile has the same form as for a Newtonian fluid. In addition, we evaluate the effective viscosity of the suspension for several representative cases in which the velocity field is Newtonian in form. Finally, as an example of the general situation in which the bulk velocity field does not remain uni-directional, we obtain the solution for the motion through a circular tube when the magnetic particles are spherical and the magnetic field is applied at right angles to the tube axis.

1. Introduction

In recent years, considerable effort has been directed toward the theoretical determination of the macroscopic, rheological properties of particulate suspensions from a consideration of the interactions, on the microscopic scale, between the suspending fluid and the individual particles when the suspension is undergoing a given bulk motion. When the suspension is dilute, the suspending fluid Newtonian, the particles rigid spheres, and the particle Reynolds number sufficiently small, the suspension can be described in bulk as a Newtonian fluid with an effective viscosity $\mu^* = \mu(1+\frac{5}{2}c)$ provided only that the particles are not

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subjected to an externally applied force or couple (cf. Einstein 1906, 1911; Batchelor 1967). For the more general case where the particles are non-spherical, or are subjected to an externally applied couple, the non-isotropic, microscopic structure of the suspension usually results in a non-Newtonian form for the bulk stress tensor (Batchelor 1970). The supposition of Newtonian behaviour, with the effect of fluid-particle interactions limited to modifications of the effective viscosity, is inadequate for the general rigid-particle suspension. It is the purpose of the present communication to investigate, for a particular class of flows, the dynamical consequences of the non-Newtonian form for the bulk stress which arises due to the action of an externally applied couple when the particles are spheroidal in shape.

There are, of course, several physical mechanisms for applying a torque to the particles of a suspension. Of these, Brenner (1970) has recently discussed the creeping motion of a suspension of neutrally buoyant, 'loaded' spheres (i.e. spheres whose centres of mass and volume are not coincident) in the presence of a gravitational field. In our work, which paralleled that of Brenner (1970), we employ Batchelor's (1970) expression for the bulk stress to deal with the similar problem of the motion of a dilute suspension of permanently magnetized spheroids in the presence of a uniform applied magnetic field. We limit our investigation to cases in which the motion of a Newtonian fluid would be uni-directional. A third possible mode for applying torque to suspension particles, which has not yet been fully investigated, involves the action of an electric (or magnetic) field on dielectric (or magnetizable) particles. The physical mechanism differs from that of the present paper in that the particle dipole is induced rather than permanent. This additional interaction with the external field causes the motion of the individual particles of the suspension to be different from that which occurs when the dipole is permanent.

After presenting the relevant rheological and dynamical equations applicable to spheroidal particles of arbitrary aspect ratio, we consider the bulk motion for two particular cases. In the first, the particles are assumed to be perfectly spherical and the field strength to be essentially arbitrary. Here we employ Hall & Busenberg (1969) in discussing the motion of the individual particles. In the second case, the particles are taken as arbitrary spheroids, but the magnetic field is assumed to be sufficiently strong to ensure that the magnetic dipole axis of each particle is effectively aligned with the field vector **H**. This division of the problem is largely a matter of convenience, which is motivated by the extremely complicated nature of the bulk rheological equation in the case of general spheroids (Batchelor 1970). First of all, the couple gives rise to both a symmetric and a non-symmetric contribution to the bulk stress. Even in the absence of any applied couples, however, the bulk stress remains non-Newtonian in form when the particles are spheroidal (though it is then symmetric). On the other hand, when the particles are spherical, the effect of an externally applied couple is confined to the generation of an anti-symmetric contribution to the bulk stress. Separate consideration of the spherical case thus allows a qualitative estimation of the role of the non-symmetric portion of the bulk stress in determining the nature of the bulk motion of a suspension in which external particle couples are present. The subsequent consideration of the case of general spheroidal particles then allows a limited, though representative, assessment of the effect of particle shape. Fortunately, the form of the anti-symmetric contribution to the bulk stress is unchanged in generalizing to the case of spheroids, so that any corresponding changes in the fundamental character of the bulk velocity fields can be wholly ascribed to the resultant non-Newtonian nature of the symmetric contributions to the bulk stress.

Although some of our results in the spherical case are similar to those of Brenner (1970), the present work concentrates on situations in which the motion of a Newtonian fluid would be uni-directional, and, in particular, presents a more detailed account of the general properties of the resulting flow fields. In so doing, it represents a necessary preliminary to the rheologically more complicated case of a suspension of spheroidal particles. The basic solution procedures we have used, though developed independently, bear a very close resemblance to those employed by Brenner (1970). In order to avoid undue repetition, we have abbreviated considerably the preliminary parts of the paper that had been prepared shortly before Brenner's work was published.

2. The expression for the bulk stress

We consider a suspension of permanently magnetized, rigid spheroids in an incompressible Newtonian fluid. We suppose that the suspended particles are close enough so that, for purposes of calculating the bulk motion, the suspension may be considered as a homogeneous fluid, and, in addition, sufficiently small that the effects of gravity and inertia may be neglected relative to viscous forces in the flow near one particle. However, the particles are assumed to be large enough for random rotations due to Brownian motion to be negligible, and to be sufficiently far apart for both hydrodynamic and magnetic dipole interactions between the particles to also be neglected. The magnetic dipole moment of the spheroids M is assumed to be fixed and the dipole axis ($\mathbf{M}/M$) to be coincident with the axis of revolution of the particles.† Furthermore, we suppose that the suspension is sufficiently dilute that the induced magnetic field can be neglected compared to the applied field. In order to avoid magnetic body forces on the particles, the applied magnetic field is assumed to be uniform (i.e. $\mathbf{H}$ = const.).

Batchelor (1970) gave the general form of the bulk stress for a suspension of spheroids with the properties described above when the particles are subjected to external body couples. Thus,

$$\sigma_{ij} = -p\delta_{ij} + 2\mu e_{ij} + \sigma'_{ij}, \tag{1}$$

where

$$\begin{aligned}\sigma'_{ij} = \frac{4\mu c}{I_1}\bigg\{&\frac{J_1}{J_2}(p_ip_j - \tfrac{1}{3}\delta_{ij})p_kp_le_{kl} + e_{ij} - p_ip_ke_{jk} - p_jp_ke_{ik}\\ &+ (p_ip_j + \tfrac{1}{3}\delta_{ij})p_kp_le_{kl} + \frac{I_1}{I_2}(p_ip_ke_{jk} + p_jp_ke_{ik} - 2p_ip_jp_kp_le_{kl})\bigg\}\\ &+ \frac{3c}{8\pi ab^2}L_k\left\{\epsilon_{ijk} + \frac{a^2-b^2}{a^2+b^2}q_k(r_ip_j + r_jp_i) - \frac{a^2-b^2}{a^2+b^2}r_k(p_iq_j + p_jq_i)\right\},\end{aligned} \tag{2}$$

† Subsequently, we shall employ $\mathbf{m}$ to denote $\mathbf{M}/M$.

and $$L_i \equiv \frac{16\kappa\mu}{3}\left\{\frac{ab^2(a^2+b^2)\,p_ip_j}{(a^2+b^2)\,J_2+a^2I_2}+\frac{ab^2(a^2+b^2)^2}{(a^2+b^2)^2J_2+2a^2b^2I_2}(\delta_{ij}-p_ip_j)\right\}\Gamma_j^{(c)}, \tag{3}$$

$$\Gamma_j^{(c)} \equiv \Gamma_j-\Gamma_j^{(s)}, \tag{4}$$

$$\Gamma_i^{(s)} \equiv \frac{a^2-b^2}{a^2+b^2}(p_jq_kr_i-p_kq_ir_j)\,e_{jk}. \tag{5}$$

The vectors $\mathbf{p}$, $\mathbf{q}$, $\mathbf{r}$ are orthogonal unit vectors, with $\mathbf{p}$ parallel to the axis of revolution of the spheroid. The semi-diameters of the spheroid, measured parallel and perpendicular to the axis of revolution, have been denoted by a and b, respectively. The non-dimensional integral functions I_i, J_i depend only on the shape of the spheroid, and are defined in Batchelor (1970).

In the above equations, p is the bulk pressure, $\mathbf{e}$ the bulk rate-of-strain tensor, c the volume concentration of spheroids in the suspension and μ the viscosity of the suspending (ambient) fluid. Also, $\mathbf{L}$ is the external torque applied to the particles, which gives rise to a rotation of the particle relative to the surrounding fluid with angular velocity $\mathbf{\Gamma}^{(c)}$. In addition, when the particles are not spherical, the pure straining portion of the motion of the ambient fluid causes an additional relative rotation of the particle with angular velocity $\mathbf{\Gamma}^{(s)}$. A full description of the origin and physical significance of the various contributions to the 'particle stress' σ'_{ij} may be found in Batchelor's (1970) paper. For the present purposes, it will suffice to emphasize the distinctly non-Newtonian nature of the bulk stress. This is particularly evident in the contribution which arises from the external couple, since it consists, in part, of an anti-symmetric term. However, for general spheroids, even the symmetric contributions to the bulk stress are non-Newtonian in form, the coefficient relating the stress to the bulk rate of strain, for example, being essentially a fourth-order tensor. It is of fundamental interest to determine the dynamical consequences of this non-Newtonian character of the bulk stress. In this paper we therefore consider the relatively simple, though representative, example of the steady motion of the suspension (described by (1)–(5)) for cases in which the motion of a Newtonian fluid would be uni-directional.

3. The equations for flow in a straight tube of constant cross-section

We begin with the usual equations of steady motion, and use the relations (1)–(5):

$$\rho(\mathbf{u}\,.\,\nabla)\,\mathbf{u} = -\nabla p+\mu\nabla^2\mathbf{u}+\nabla\,.\,\boldsymbol{\sigma}'+O(c^2), \tag{6a}$$

$$\nabla\,.\,\mathbf{u} = 0. \tag{6b}$$

We note that $\nabla\,.\,\boldsymbol{\sigma}'$ is of $O(c)$. As indicated in (6*a*), these equations are only valid to $O(c)$, where $c \ll 1$, since particle-particle interactions were neglected in the derivation of the constitutive relation equation (2). Hence, in seeking a solution for the bulk motion in a given situation, we employ the expansions,

$$\left.\begin{aligned} \mathbf{u} &= \mathbf{u}^{(0)}+c\mathbf{u}^{(1)}+O(c^2),\\ p &= p_0+cp_1+O(c^2),\\ \frac{1}{c}\boldsymbol{\sigma}' &= \boldsymbol{\sigma}'^{(0)}+c\boldsymbol{\sigma}'^{(1)}+O(c^2). \end{aligned}\right\} \tag{7}$$

Since the lowest-order approximation to the particle stress is $O(c)$, it is clear, in view of the accuracy of (6 a), that only $\boldsymbol{\sigma}'^{(0)}$ will enter into the analysis. Formally, $\boldsymbol{\sigma}'^{(0)}$ is obtained from (2) to (5) by employing the dynamical variables e_{ij}, L_i, ω_i and Γ_i, evaluated using the $O(1)$ velocity field $\mathbf{u}^{(0)}$. Throughout the remainder of this paper, we shall assume that the undisturbed velocity field corresponds to the uni-directional, fully developed flow of a Newtonian fluid through a straight cylindrical tube, and hence that $\mathbf{u}^{(0)}$ (and thus $\boldsymbol{\sigma}'^{(0)}$ also) is a known function satisfying

$$\mu\nabla^2\mathbf{u}^{(0)} = \text{const.}, \quad \nabla . \mathbf{u}^{(0)} = 0.$$

Therefore, we consider here only the determination of the $O(c)$ modification which results from the presence of the particles, i.e. $\mathbf{u}^{(1)}$ and p_1. The general equations governing the $O(c)$ velocity and pressure fields for fully developed flow in a tube are simply

$$\rho(\mathbf{u}^{(1)}.\nabla)\,\mathbf{u}^{(0)} = -\nabla p_1 + \mu\nabla^2\mathbf{u}^{(1)} + \nabla . \boldsymbol{\sigma}'^{(0)} \qquad (8\,a)$$

$$\nabla . \mathbf{u}^{(1)} = 0. \qquad (8\,b)$$

The term $(\mathbf{u}^{(0)}.\nabla)\,\mathbf{u}^{(1)}$, which would generally appear in (8 a), is identically zero here as a result of the assumption that the velocity field is fully developed, i.e. that $\mathbf{u}$ $(= \mathbf{u}^{(0)} + c\mathbf{u}^{(1)} + \ldots)$ is independent of position along the direction of the tube axis. A further consequence of this assumption is that the components of (8 a), governing motion in the plane normal to the direction of the undisturbed flow, are uncoupled from the equation governing the velocity component parallel to $\mathbf{u}^{(0)}$.† Combined with their linearity in $\mathbf{u}^{(1)}$, this uncoupling renders the solution of (8 a, b) particularly simple in principle. Indeed, if we denote the $O(1)$ and $O(c)$ velocity components in the direction of the undisturbed motion as w_0 and w_1, respectively, and define the ordinary stream function ψ based on the velocity components in the plane normal to $\mathbf{u}^{(0)}$, (8 a, b) can be expressed simply as

$$\nabla^4\psi = i.[\nabla \times (\nabla . \boldsymbol{\sigma}'^{(0)})], \qquad (9\,a)$$

$$\mu\nabla^2 w_1 = i.[\nabla p_1 + \rho(\mathbf{u}^{(1)}.\nabla)\,\mathbf{u}^{(0)} - \nabla . \boldsymbol{\sigma}'^{(0)}]. \qquad (9\,b)$$

Here i denotes the unit vector in the direction of the undisturbed velocity $\mathbf{u}^{(0)}$. These equations are to be solved subject to the usual no-slip condition at solid surfaces of the flow vessel,

$$\mathbf{u}^{(1)} = 0 \quad \text{on} \quad S. \qquad (9\,c)$$

In the form (9 a–c), it is evidently possible that the resultant velocity field may be neither uni-directional, nor have a Newtonian form for the streamwise profile (i.e. $w_0 + cw_1$). We note from (8 a), however, that

$$\nabla \times (\nabla . \boldsymbol{\sigma}'^{(0)}) \equiv 0 \qquad (10)$$

is both a necessary and a sufficient condition for the *form* of the velocity field to be unchanged (to $O(c)$) from that which would occur with a Newtonian fluid.

† We note, in passing, that the assumption of fully developed, uni-directional undisturbed flow also leads to considerable simplification in the term $\nabla . \boldsymbol{\sigma}'^{(0)}$ which we will discuss at a later stage of the paper.

Provided that the velocity field is fully developed, it can be shown (as is obvious from $9a, b$) that an entirely equivalent set of conditions is

$$i.[\nabla \times (\nabla . \boldsymbol{\sigma}'^{(0)})] \equiv 0, \tag{11a}$$

$$i.[\nabla . \boldsymbol{\sigma}'^{(0)}] = \text{const.} \tag{11b}$$

If $(11a)$ is satisfied, then there will be no motion in the plane normal to $\mathbf{u}^{(0)}$. If, in addition, $(11b)$ is satisfied, then the resultant uni-directional velocity field will have a profile which is completely indistinguishable from that of a Newtonian fluid with an effective viscosity,

$$\mu^* = \mu\left(1+\left\{\frac{i.[\nabla . \boldsymbol{\sigma}'^{(0)}]}{\mu\nabla^2\mathbf{u}^{(0)}}\right\}c\right). \tag{12}$$

Hence, under the rather special circumstances in which *both* of $(11a, b)$ (or equivalently (10)) are met, it is sufficient, so far as the velocity field is concerned, to consider the rheological effects of the particle shape and external couples solely in terms of variations of the effective viscosity. In view of $(9a, b)$, however, the concept of Newtonian behaviour with a modified effective viscosity is insufficient in the general case, even for a determination of the bulk velocity field.

In §§ 4–6 we consider the application of the general equations of § 3 to the special cases of (i) spherical particles with magnetic fields of arbitrary strength, and (ii) general spheroidal particles in the strong-field limit. Rather than simply solve $(9a–c)$ for a variety of vessel geometries, as is possible, we concentrate attention on the development of general criteria applicable to the whole class of problems, which will allow an *a priori* assessment of the nature of the motion that will occur in any particular case.

4. General results for spherical particles

A considerable simplification occurs in the equations of § 3 when the particles are assumed to be spherical. In particular, (1)–(5) reduce to

$$\sigma_{ij} = -p\delta_{ij} + 2\mu(1+\tfrac{5}{2}c)\, e_{ij} + 3\mu c\epsilon_{ijk}\, \Gamma_k, \tag{13}$$

so that the $O(c)$ equations of motion $(9a, b)$ become

$$\nabla^4\psi = i.[\nabla \times (\nabla \times \boldsymbol{\Gamma}^{(0)})], \tag{14a}$$

$$\mu\nabla^2 w_1 = i.[\nabla p_1 + \rho(\mathbf{u}^{(1)}.\nabla)\,\mathbf{u}^{(0)} - \tfrac{5}{2}\mu\nabla^2\,\mathbf{u}^{(0)} - 3\mu\nabla \times \boldsymbol{\Gamma}^{(0)}]. \tag{14b}$$

$\boldsymbol{\Gamma}^{(0)}$ denotes the angular velocity of the spherical particles relative to the $O(1)$ velocity field (i.e. relative to $\frac{1}{2}\,\text{curl}\,\mathbf{u}^{(0)}$). Brenner (1970) evaluated $\boldsymbol{\Gamma}^{(0)}$ in terms of the undisturbed flow field and the particle dipole vector. For the case of permanently magnetized spheres in a magnetic field, his expression becomes

$$\boldsymbol{\Gamma}^{(0)} \equiv -\tfrac{1}{2}\boldsymbol{\omega}^{(0)} + \{\tfrac{1}{2}\boldsymbol{\omega}^{(0)}.\mathbf{m}\}\,\mathbf{m}, \tag{15}$$

where $\mathbf{M}$ is the magnetization vector ($M = |\mathbf{M}|$ = dipole strength) and $\boldsymbol{\omega}^{(0)} \equiv \text{curl}\,\mathbf{u}^{(0)}$. The validity of this expression depends crucially on the fact (established by Hall & Busenberg 1969) that $\mathbf{m}$ is fixed in space. These authors employed the standard Poincare–Bendixson theory to show that the solution of the equations for the creeping motion of an individual magnetic sphere, immersed in a simple shear flow in the presence of a uniform external field, has a globally stable stationary point corresponding to free rotation of the particle about its magnetic dipole axis, which assumes a *fixed* orientation relative to the magnetic field vector **H** and the undisturbed vorticity vector $\boldsymbol{\omega}^{(0)}$. This fixed orientation is defined by

$$\mathbf{m} = \cos\phi_S\left(\frac{\boldsymbol{\omega}^{(0)}}{\omega^{(0)}}\right) + \sin\phi_S\cos\theta_S\left[\frac{(\boldsymbol{\omega}^{(0)}\times\mathbf{H})\times\boldsymbol{\omega}^{(0)}}{|(\boldsymbol{\omega}^{(0)}\times\mathbf{H})\times\boldsymbol{\omega}^{(0)}|}\right] + \sin\phi_S\sin\theta_S\left[\frac{\boldsymbol{\omega}^{(0)}\times\mathbf{H}}{|\boldsymbol{\omega}^{(0)}\times\mathbf{H}|}\right], \tag{16}$$

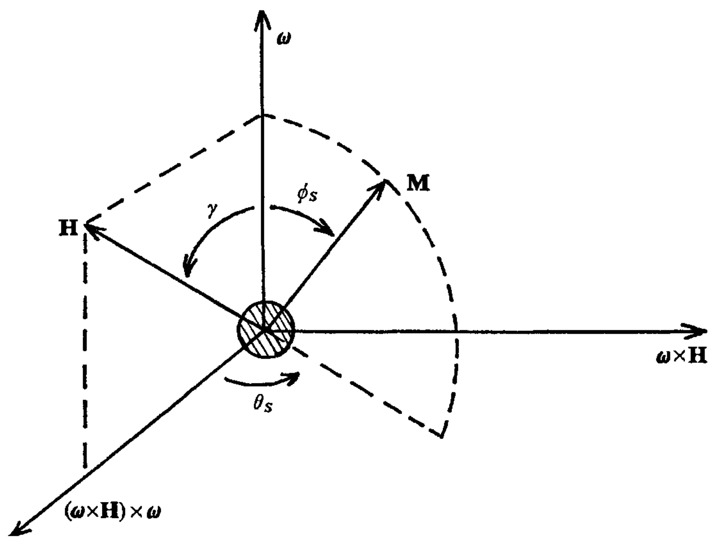

FIGURE 1. The intrinsic co-ordinate system employed by Hall & Busenberg (1969).

where
$$\sin^2\phi_S = \tfrac{1}{2}(1+\beta^{-2}) - [\tfrac{1}{4}(1+\beta^{-2})^2 - \beta^{-2}\sin^2\gamma]^{\frac{1}{2}}, \tag{17a}$$
$$\sin\theta_S = \beta\csc\gamma\sin\phi_S, \tag{17b}$$

together with the additional requirements that ϕ_S and γ lie in the same quadrant and $0 < \theta_S < \frac{1}{2}\pi$. Here γ is the angle between **H** and $\boldsymbol{\omega}^{(0)}$ and the parameter β is defined as
$$\beta \equiv \frac{4\pi\mu a^3\omega^{(0)}}{\kappa MH}, \tag{18}$$

in which a is the particle radius and κ is the magnetic permeability of the suspending fluid. As shown in figure 1, ϕ_S and θ_S are the polar angles in an intrinsic co-ordinate system defined relative to the vectors $\boldsymbol{\omega}^{(0)}$ and **H**. An exceptional case, discussed by Brenner (1970), for which this stationary point is *not* stable, occurs when $\gamma = \frac{1}{2}\pi$ and $\beta > 1$. In what follows, we assume that if $\gamma = \frac{1}{2}\pi$, then $\beta \leqslant 1$; otherwise we do not restrict the magnetic field strength. The equations (14)–(18) are sufficient, in principle, to determine the $O(c)$ modification of the bulk velocity field in the suspension, given the details of **H** and the geometry of the flow region (hence $\mathbf{u}^{(0)}$). It is worth noting that a considerable simplification occurs in the strong-field limit where $\beta \to 0$, since then **M** and **H** are effectively parallel, and (16)–(18) become superfluous; the particles simply rotate about the **H** axis.

Brenner (1970) noted the important conclusion (which at first seems surprising, in view of the distinctly non-Newtonian form of the stress tensor (13)), that, when

$$\nabla \times (\nabla \times \mathbf{\Gamma}^{(0)}) \equiv 0, \tag{19}$$

the velocity field (to $O(c)$) will be indistinguishable from that for a Newtonian fluid. This condition (19) is, of course, a special case (for spherical particles) of the general condition (10) which we noted previously, and is applicable to arbitrary undisturbed velocity fields. Nevertheless, we consider the alternative and entirely equivalent set of conditions corresponding to (11 *a*, *b*), which have the following two obvious advantages for application to cases in which the undisturbed motion is uni-directional. First, it is important to be able to predict whether secondary motion occurs (regardless of the form of the 'uni-directional' velocity component). This follows from the fact that the actual computation of ψ can be extremely tedious, owing to the often complicated form of the right-hand side of (14 *a*), coupled with the observation that, in general, this solution is a necessary prerequisite for the determination of w_1, which is the quantity of primary interest since it is directly related to such important properties as the volume flow-rate. Secondly, the condition (11 *b*) is of a form which, when satisfied, allows immediate determination of the effective viscosity of the suspension. In addition to these inherent advantages, we expand the relevant vector equations to obtain conditions expressed wholly in terms of the properties of the applied field **H** and the undisturbed velocity field $\mathbf{u}^{(0)}$.

4.1. *The existence of secondary flow*

From (11 *a*) and (13), the condition that no secondary motion should occur is simply

$$i \,.\, [\nabla \times (\nabla \times \mathbf{\Gamma}^{(0)})] \equiv 0. \tag{20a}$$

This expression is the only non-homogeneous source term for the vorticity component $i \,.\, \boldsymbol{\omega}^{(1)}$, which must itself be non-zero in general, for secondary motion to occur. On employing the definition of $\mathbf{\Gamma}^{(0)}$, equation (15), the condition (20*a*) can be expressed as

$$i \,.\, [\tfrac{1}{2}\nabla^2\boldsymbol{\omega}^{(0)} + \nabla[\nabla \,.\, \{(\tfrac{1}{2}\boldsymbol{\omega}^{(0)} \,.\, \mathbf{m})\,\mathbf{m}\}] - \nabla^2\{(\tfrac{1}{2}\boldsymbol{\omega}^{(0)} \,.\, \mathbf{m})\,\mathbf{m}\}] \equiv 0.$$

Since the undisturbed uni-directional motion has been assumed to be fully developed, and since $i \,.\, \boldsymbol{\omega}^{(0)} \equiv 0$, this simplifies to

$$i \,.\, [\nabla^2\{\tfrac{1}{2}\boldsymbol{\omega}^{(0)} \,.\, \mathbf{m}\}\,\mathbf{m}] \equiv 0. \tag{20b}$$

Taken in conjunction with the expression (16) for $\mathbf{m}$, this condition is still quite complex. In two important particular cases, however, no secondary motion will occur. First, if the applied field **H** is aligned parallel to the uni-directional undisturbed motion, then $\gamma = \tfrac{1}{2}\pi$, and, provided $\beta \leqslant 1$ so that (16) is valid,

$$\boldsymbol{\omega}^{(0)} \,.\, \mathbf{m} \equiv 0.$$

Since (20 *b*) is linear in $\boldsymbol{\omega}^{(0)} \,.\, \mathbf{m}$, it is clear that no secondary motion can occur in this case. The second class of problems, where (20 *b*) is satisfied, occurs, for *arbitrary* geometry of the flow region and direction of the magnetic field, in the strong-field limit $\beta = 0$, where $\mathbf{m}$ and $\mathbf{H}$ are effectively parallel. Provided that $\mathbf{H}$ is uniform, as assumed, (20*b*) reduces to

$$i \cdot \left[\frac{\mathbf{H}}{H}\left(\frac{\mathbf{H}}{H} \cdot \nabla^2 \boldsymbol{\omega}^{(0)}\right)\right] \equiv 0 \qquad (20c)$$

as $\beta \to 0$. Since $\nabla^2 \boldsymbol{\omega}^{(0)} \equiv 0$ for arbitrary (undisturbed) uni-directional flows, we conclude that, provided the magnetic field is strong enough to align the particle dipoles with $\mathbf{H}$, there can be no secondary flow. In a magnetic suspension consisting of single domain ferromagnetic particles of cobalt in toluene, McTague (1969) has estimated that this strong field condition is satisfied for values of H larger than about 1000 Gauss, a moderate value.

4.2. *The form of the velocity profile for cases of no secondary motion*

The determination of the form of the velocity component w_1 (14*b*) is complicated in the general case by the coupling of the inertial terms with the secondary velocity field. When no secondary motion occurs, however, the condition (11*b*) is sufficient for the modified velocity profile to have the same form as would occur for a Newtonian fluid. When the particles are spherical, this condition can be simply expressed as

$$i \cdot [3\mu \nabla \times \boldsymbol{\Gamma}^{(0)}] = K\mu f(\alpha, \delta). \qquad (21a)$$

Here we have employed K to denote the constant $i \cdot \nabla^2 \mathbf{u}^{(0)}$. As shown in figure 2, the angles α and δ define the orientation of the magnetic field relative to the undisturbed uni-directional velocity, $\mathbf{u}^{(0)}$. When (21) (as well as (20)) is satisfied, the velocity and pressure fields can, as stated previously, be calculated by treating the suspension as a Newtonian fluid with an effective viscosity (see (12)) of

$$\mu^* = \mu(1 + \tfrac{5}{2}c + f(\alpha, \delta)\, c). \qquad (22)$$

It is clear from (21) and (22) that μ^* depends not only on the orientation of the magnetic field $\mathbf{H}$ but also on the form of the undisturbed velocity profile, and hence on the geometry of the flow vessel.

The condition (21) is trivially satisfied with $f(\alpha, \delta) \equiv \frac{3}{2}$ when $\mathbf{H}$ is parallel to $\mathbf{u}^{(0)}$, since $\boldsymbol{\Gamma}^{(0)} \equiv -\frac{1}{2}\boldsymbol{\omega}^{(0)}$ for all β $(0 \leqslant \beta \leqslant 1)$. Hence, as noted by Brenner (1970), the modified velocity field will be of the same form as that of a Newtonian fluid with viscosity $\mu^* = \mu(1 + 4c)$. We note that the effective viscosity in this case has the unusual property of being completely independent of the vessel geometry ($\mathbf{u}^{(0)}$).

In the strong-field case, where β is effectively zero, the condition (21) becomes

$$i \cdot \left[\tfrac{3}{2}\mu\left(\nabla^2 \mathbf{u}^{(0)} \cdot \frac{\mathbf{H}}{H}\right)\frac{\mathbf{H}}{H} + \tfrac{3}{2}\mu\left\{\nabla \times \left(\tfrac{1}{2}\boldsymbol{\omega}^{(0)} \times \frac{\mathbf{H}}{H}\right)\right\} \times \frac{\mathbf{H}}{H}\right] = K\mu f(\alpha, \delta), \qquad (21b)$$

provided $\mathbf{H}$ is uniform. In order to obtain explicit requirements on the form of the undisturbed flow which satisfies this condition, it is convenient to employ the Cartesian co-ordinate system we have already defined in figure 2. The direction of undisturbed motion is taken parallel to $\mathbf{i}_3$, with the $\mathbf{i}_1$, $\mathbf{i}_2$ axes being arbitrary, except for the requirement that $(\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3)$ be orthogonal. We take

$$\mathbf{u}^{(0)} = u^{(0)}(x_1, x_2)\, \mathbf{i}_3,$$

$$\boldsymbol{\omega}^{(0)} \equiv \operatorname{curl} \mathbf{u}^{(0)} = \omega_1^{(0)}(x_1, x_2)\,\mathbf{i}_1 + \omega_2^{(0)}(x_1, x_2)\,\mathbf{i}_2,$$

and let
$$\mathbf{H} = H(\sin\alpha \mathbf{i}_3 + \cos\alpha\cos\delta \mathbf{i}_1 + \cos\alpha\sin\delta \mathbf{i}_2).$$

With these conventions it is easily shown that (21 *b*) becomes

$$\tfrac{3}{2}\mu K \sin^2\alpha + \tfrac{3}{2}\mu\cos^2\alpha\left[\sin\delta\cos\delta\left(\frac{\partial\omega_1^{(0)}}{\partial x_1} - \frac{\partial\omega_2^{(0)}}{\partial x_2}\right) + \left(\sin^2\delta\frac{\partial\omega_1^{(0)}}{\partial x_2} - \cos^2\delta\frac{\partial\omega_2^{(0)}}{\partial x_1}\right)\right] \equiv \mu f(\alpha,\delta)K. \quad (21\,c)$$

The first-term on the left-hand side clearly contributes a 'Newtonian' modification to the undisturbed uni-directional velocity profile. The general condition (21 *a*) thus reduces, in the special case $\beta \sim 0$, to the requirement that both $\omega_1^{(0)}$ and $\omega_2^{(0)}$ be either constant or linear functions of x_1 and x_2. This will be the case for all two-dimensional uni-directional flows, as well as such common three-

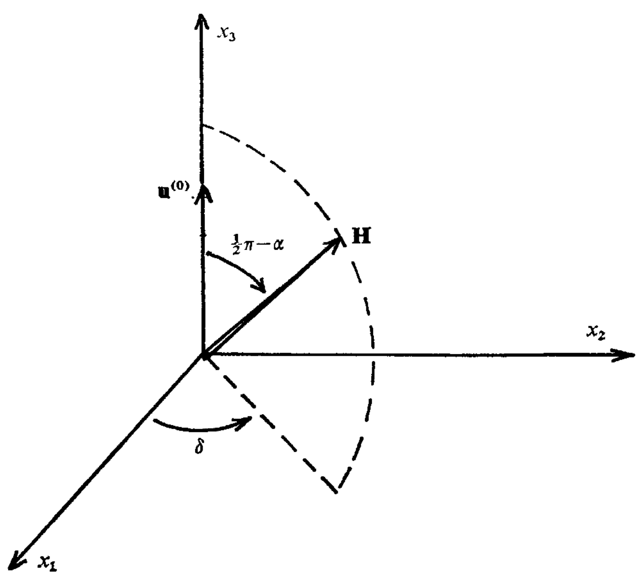

FIGURE 2. The co-ordinate axes relative to the undisturbed velocity vector and the magnetic field vector.

dimensional situations as the flow through tubes of circular and elliptic cross-section, and hence the modified velocity field will be of the same form as that of a Newtonian fluid in these cases, provided only that $\beta \sim 0$. As indicated by (22), the evaluation of $f(\alpha, \delta)$ from (21 *c*) is equivalent to finding the effective viscosity μ^*. For reference, we list $(\mu^* - \mu)/\mu c$ for the circular and elliptic tube and for two-dimensional Poiseuille flow in table 1, where, for convenience, we have taken $\mathbf{i}_1$, $\mathbf{i}_2$ to be coincident with the axes of symmetry of the undisturbed velocity profiles. For the special case of two-dimensional Poiseuille flow with the magnetic field vector **H** normal to the vorticity vector this result was obtained previously by Hall & Busenberg (1969), while that for a circular tube oriented with its axis parallel or perpendicular to **H** was obtained by Brenner (1970). The flow through an annulus of arbitrary cross-section is an example of a class of problems in which the vorticity condition is not met, and hence, though the velocity field remains uni-directional according to (20c), the profile will differ in form from that of a Newtonian fluid. This serves to emphasize that it is only under quite special circumstances that the suspension will have a Newtonian form of the velocity distribution.

Geometry	$(\mu^*-\mu)/(\mu c)$
(i) *Circular tube*	$\frac{5}{2}+\frac{3}{2}(1-\frac{1}{2}\cos^2\alpha)$
(ii) *Infinite parallel walls* (two-dimensional Poiseuille flow). Note: $\delta = \frac{1}{2}\pi$ corresponds to **H** being in the *plane* of shear flow	$\frac{5}{2}+\frac{3}{2}\sin^2\alpha[1+\cot^2\alpha\sin^2\delta]$
(iii) *Elliptic tube* (with surface $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$). Note: $\delta = \frac{1}{2}\pi$ corresponds to **H** being coincident with the major semi-diameter a	$\frac{5}{2}+\frac{3}{2}\sin^2\alpha+\frac{3}{2}\cos^2\alpha\left(\frac{a^2b^2}{a^2+b^2}\right)\times\left(\frac{\sin^2\delta}{b^2}+\frac{\cos^2\delta}{a^2}\right)$

TABLE 1

We note, as might be expected, that the class of uni-directional undisturbed flows, for which the resultant velocity field is completely Newtonian (hence satisfying Brenner's condition (19) or equivalently our (20) and (21)), is demonstrably smaller than that for which the motion simply remains uni-directional. In practical terms, this distinction is quite important, since it implies that the detailed analysis of the velocity and pressure fields is greatly simplified for a significantly more general class of problems than one would have expected on the basis of the condition (19). With regard to the latter cases, it is noteworthy that the very suggestive result obtained by Brenner (1970), whereby an apparent viscosity of $\mu(1+\frac{13}{4}c)$ was found *both* for flow through a horizontal circular tube (equivalent to $\alpha = 0$) and for horizontal Stokes' translation of a large sphere, is not recovered for the flow through a tube of elliptic cross-section, nor would it be expected, except by chance, in other more general flow configurations.

5. Flow of a suspension of magnetic spheres through a circular tube

The laminar flow through a circular tube provides a particularly simple and interesting example, which illustrates many of the general properties outlined in § 4.

We employ cylindrical co-ordinates (ρ, ϕ, z), as shown in figure 3, with the z axis in the direction of the undisturbed uni-directional flow. Due to the independence of the tube geometry on ϕ, the applied magnetic field **H** is characterized by a single angle α. If we choose the (y, z) plane so that it contains **H**, then we can write

$$\mathbf{H} = H(\cos\alpha\sin\phi\,\mathbf{i}_\rho+\cos\alpha\cos\phi\,\mathbf{i}_\phi+\sin\alpha\,\mathbf{i}_z).$$

The applied pressure gradient along the pipe is supposed to be given, equal to $-P$, i.e.

$$-\frac{\partial p}{\partial z} = P \quad \text{(constant)},$$

and the velocity field to be fully developed. Then the 'undisturbed' velocity field, $\mathbf{u}^{(0)}$, consists of simple Poiseuille flow,

$$u_z^{(0)} = 1-\rho^2, \quad u_\phi^{(0)} = u_\rho^{(0)} = 0,$$

where we have non-dimensionalized with respect to the centreline velocity $w \equiv (Pa^2)/(4\mu^*)$ and the tube radius. For simplicity, we have included the Einstein contribution to the velocity profile in the zeroth solution.

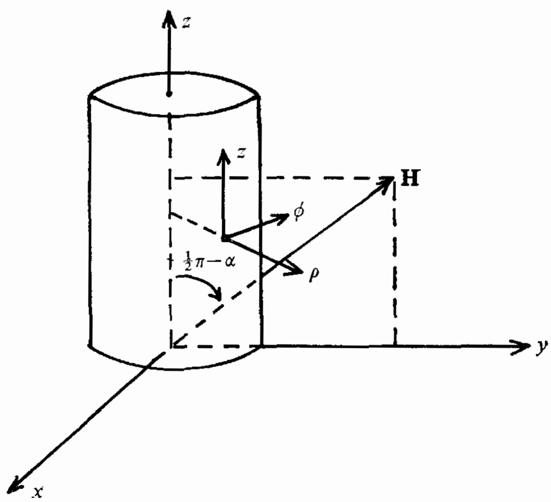

FIGURE 3. The co-ordinate system for the flow through a circular cylinder.

As noted in §4, the $O(c)$ equations for $u_\rho^{(1)}$ and $u_\phi^{(1)}$ are independent of $u_z^{(1)}$. Hence, we introduce the stream function ψ, defined by

$$u_1 \equiv -\frac{1}{\rho}\frac{\partial \psi}{\partial \phi}, \quad v_1 \equiv \frac{\partial \psi}{\partial \rho},$$

so that the governing equations ($14a, b$), minus the Einstein term, become

$$\nabla^4\psi = 3\rho\left[\frac{\partial^2\Gamma_z^{(0)}}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial\Gamma_z^{(0)}}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2\Gamma_z^{(0)}}{\partial\phi^2}\right], \tag{23a}$$

$$\nabla^2 w_1 = R\frac{\partial w_0}{\partial\rho}u_1 + 3\left[\frac{1}{\rho}\frac{\partial\Gamma_\rho^{(0)}}{\partial\phi} - \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho\Gamma_\phi^{(0)})\right]. \tag{23b}$$

Here, for convenience, we have employed (u, v, w) to denote (u_ρ, u_ϕ, u_z), and have called (aw/ν^*) the Reynolds number, R. We shall solve these equations subject to the boundary conditions,

$$\psi = \frac{\partial\psi}{\partial\rho} = 0, \quad w_1 = 0 \quad \text{at} \quad \rho = 1, \tag{23c}$$

and boundedness, of the solutions at the centreline, $\rho = 0$.

5.1. *The case* $\beta \equiv 0 (H \to \infty)$

The simplest case to analyse occurs for $\beta \equiv 0$, when $\mathbf{m}$ and $\mathbf{H}$ are coincident since here we can replace $\mathbf{m}$ by $\mathbf{H}/H$ in the definition of (15). Hence,

$$\mathbf{\Gamma}^{(0)} \equiv \rho \sin\alpha \cos\alpha \cos\phi \mathbf{i}_z + \rho \cos^2\alpha \cos\phi \sin\phi \mathbf{i}_\rho + \rho(\cos^2\alpha \cos^2\phi - 1)\,\mathbf{i}_\phi,$$

and the $O(c)$ equations of motion (23 a, b) become

$$\nabla^4 \psi \equiv 0,$$

$$\nabla^2 w_1 = 6(1 - \tfrac{1}{2}\cos^2\alpha).$$

The solution of these equations satisfying the boundary conditions (23 c) is

$$\psi \equiv 0,$$

$$w_1 = -\tfrac{3}{2}(1 - \tfrac{1}{2}\cos^2\alpha)(1-\rho^2).$$

Hence, as predicted by the general considerations of §4 there is no secondary motion for any angle α when $\beta = 0$. In addition, the uni-directional velocity profile remains parabolic to $O(c)$, becoming

$$u_z \equiv [1 - \tfrac{3}{2}c(1 - \tfrac{1}{2}\cos^2\alpha)]\,(1-\rho^2) + O(c^2).$$

5.2. *The case* β *small*, $\alpha = 0$.

The more general case where β is small, but non-zero, is somewhat more difficult since $\mathbf{M}$ and $\mathbf{H}$ are not necessarily coincident. However, the solution is illustrative of the resulting velocity fields when the anti-symmetric character of the stress tensor (13) is manifested in both secondary motion and a modified profile shape for the axial component.

Employing the definitions of (15)–(18), together with the relationship

$$\cos\gamma = \cos\alpha \cos\phi \tag{24}$$

between the angles γ and α, we can express $\mathbf{m}$ and hence $\mathbf{\Gamma}^{(0)}$ in terms of the components in cylindrical co-ordinates, obtaining

$$\mathbf{\Gamma}^{(0)} = -\rho \sin^2\phi_S\, \mathbf{i}_\phi + \left[\frac{\rho \cos\phi_S \sin\phi_S(\sin\theta_S \sin\alpha + \cos\theta_S \cos\alpha \sin\phi)}{(\sin^2\alpha + \cos^2\alpha \sin^2\phi)^{\frac{1}{2}}}\right]\mathbf{i}_\rho + \left[\frac{\rho \cos\phi_S \sin\phi_S(\cos\theta_S \sin\alpha - \sin\theta_S \cos\alpha \sin\phi)}{(\sin^2\alpha + \cos^2\alpha \sin^2\phi)^{\frac{1}{2}}}\right]\mathbf{i}_z.$$

The parameter β, defined in (18), which, in part, determines the angles ϕ_S and θ_S (see (17 a, b)) is conveniently expressed in terms of a new parameter $\hat{\beta}$ as

$$\beta \equiv \left[\frac{8\kappa\mu a^3}{2xMH}\left(\frac{w}{a}\right)\right]\frac{\partial w_0}{\partial \rho} \equiv \frac{\hat{\beta}}{2}\frac{\partial w_0}{\partial \rho},$$

so that $\hat{\beta}$ is characteristic of the system as a whole, being independent of the detailed motion, and thus of spatial position.

Using the definition of $\mathbf{\Gamma}^{(0)}$ above, it would clearly be possible at this point to solve the equations of motion (23 a, b) numerically for arbitrary α and $\hat{\beta} \leqslant 1$.

However, for simplicity, we have instead elected to limit our investigation to the special case where $\alpha = 0$; that is, to the case where $\mathbf{H}$ is perpendicular to the tube axis. The case $\alpha = \frac{1}{2}\pi$ was essentially solved in § 4, as well as by Brenner (1970).

Setting $\alpha = 0$, we note from (24) that $\gamma \equiv \phi$. Hence, from (17 a)

$$\sin^2 \phi_S \equiv \tfrac{1}{2}(1+\beta^{-2}) - [\tfrac{1}{4}(1+\beta^{-2})^2 - \beta^{-2}\sin^2\phi]^{\frac{1}{2}}$$

and from this, together with the expression for $\mathbf{\Gamma}^{(0)}$ given above, we obtain

$$(\Gamma^{(0)}_\rho, \Gamma^{(0)}_\phi, \Gamma^{(0)}_z) = (\tfrac{1}{2}\omega^{(0)}\cos\phi_S \sin\phi_S \cos\theta_S, -\tfrac{1}{2}\omega^{(0)}\sin^2\phi_S, -\tfrac{1}{2}\omega^{(0)}\sin\phi_S\cos\phi_S\sin\theta_S).$$

Given these expressions for $\mathbf{\Gamma}^{(0)}$, it is clearly a straightforward process to obtain an expansion for w_1 and ψ, valid for $\hat{\beta} \ll 1$.

Thus, on writing

$$\psi = \hat{\beta}\psi^{(1)} + \hat{\beta}^2\psi^{(2)} + \hat{\beta}^3\psi^{(3)} + \ldots,$$

and

$$w_1 = w_1^{(0)} + \hat{\beta}w_1^{(1)} + \hat{\beta}^2 w_1^{(2)} + \hat{\beta}^3 w_1^{(3)} + \ldots,$$

we obtain, after some lengthy algebraic manipulation, the solution

$$\psi = \tfrac{1}{8}\hat{\beta}'^3[\tfrac{1}{9800}(6\rho^7 - 21\rho^2 + 15) + \tfrac{1}{420}(2\rho^7 - 5\rho^4 + 3\rho^2)\sin 2\phi + \tfrac{1}{396}(2\rho^7 - 3\rho^6 + \rho^4)(\sin 4\phi - \tfrac{1}{2}\cos 4\phi)] + O(\hat{\beta}^5),$$

$$w_1 = -\tfrac{3}{4}(1-\rho^2) + \tfrac{1}{4}\hat{\beta}^2[\tfrac{3}{32}(1-\rho^4) + \tfrac{1}{8}\rho^2(1-\rho^2)\cos 2\phi] + \tfrac{1}{8}\hat{\beta}^3[\tfrac{1}{105}R(\tfrac{2}{77}\rho^9 - \tfrac{5}{32}\rho^6 + \tfrac{1}{4}\rho^4 - \tfrac{295}{2364}\rho^2)\cos 2\phi + \tfrac{2}{99}R(\tfrac{2}{65}\rho^9 - \tfrac{1}{10}\rho^8 + \tfrac{1}{20}\rho^6 - \tfrac{19}{1040}\rho^4)(\cos 4\phi + \tfrac{1}{2}\sin 4\phi)] + O(\hat{\beta}^4).$$

The streamlines of the flow in the transverse plane are plotted in figure 4.

In contrast to the cases in which $\beta = 0$ or $\alpha = \frac{1}{2}\pi$, this solution exhibits both secondary motion and a non-parabolic axial profile. This result is significant in that it offers analytic confirmation of the conjecture, based on the forms of (14 a, b), that the resultant velocity fields will generally be distinctly different from those which would occur if the suspension behaved as a Newtonian fluid.

6. General results for spheroidal particles

In § 4 and § 5 we discussed the bulk motion of the suspension for the special case in which the particles are spherical. In that case, we were able to establish the important conclusion that secondary flow never occurs when the strong-field approximation is valid. It is of fundamental interest to know whether this behaviour is of general application, or whether it is, perhaps, a unique consequence of the strong symmetry of spheres. In addition, it is of practical importance to determine the influence of particle shape on such overall properties as the volume flow-rate of the suspension for a given applied pressure gradient. Therefore, in this section, we consider the bulk motion of a similar suspension of spheroidal particles in the strong-field limit. For simplicity, we assume that the permanent magnetic dipole of the particles is aligned along the axis of revolution. Comparison of the results with those obtained for spheres allows at least a limited assessment of the importance of particle shape in determining the nature of the bulk motion of a suspension in which external particle couples are present.

We employ the equations of § 2, together with the strong-field definition of $\mathbf{\Gamma}^{(0)}$,

$$\mathbf{\Gamma}^{(0)} \equiv -\tfrac{1}{2}\boldsymbol{\omega}^{(0)} + \left\{\tfrac{1}{2}\boldsymbol{\omega}^{(0)}\cdot\frac{\mathbf{H}}{H}\right\}\frac{\mathbf{H}}{H}. \tag{25}$$

The physical interpretation of this definition is that the particle simply rotates freely about its dipole axis, and that this axis is co-linear with the applied field vector $\mathbf{H}$ for sufficiently strong magnetic fields. For convenience, we employ

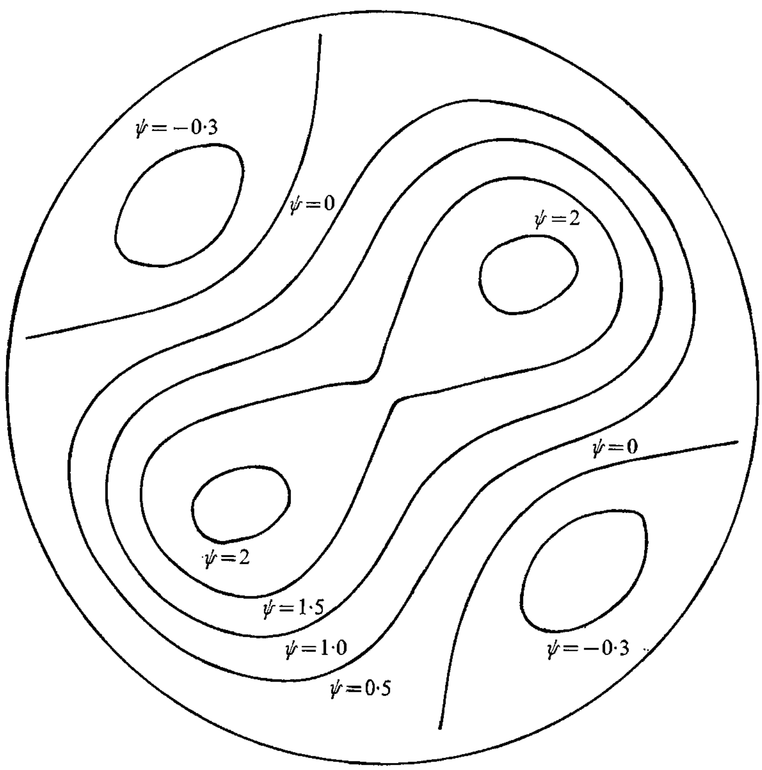

FIGURE 4. The streamline pattern for the secondary motion in the flow through a circular cylinder. (The stream-function values have been multiplied by $8\beta^{-3} \times 10^4$.)

a Cartesian co-ordinate system (see figure 5) with one axis aligned in the direction of the undisturbed, uni-directional flow. The x_1 and x_2 axes will be temporarily left unspecified. Owing to the symmetry of the spheroidal particles, the unit vectors $\mathbf{p}$, $\mathbf{q}$ and $\mathbf{r}$ may be expressed, relative to this co-ordinate system, simply in terms of the polar angles θ and ϕ of the axis of revolution of the particles as

$$\left.\begin{aligned} \mathbf{p} &= (\cos\theta, \sin\theta\cos\phi, \sin\theta\sin\phi), \\ \mathbf{q} &= (-\sin\theta, \cos\theta\cos\phi, \cos\theta\sin\phi), \\ \mathbf{r} &= (0, -\sin\phi, \cos\phi). \end{aligned}\right\} \tag{26}$$

In the strong-field limit which we have assumed, the magnetic field vector is co-linear with the particle dipole, which has itself been assumed to coincide with the axis of revolution of the particle. Hence, θ and ϕ define the orientation of the applied field (**H**) relative to the (x_1, x_2, x_3) co-ordinate system, and can thus be considered to be given in the specification of the problem in a particular situation. This constitutes the major simplification which occurs because of the strong-field approximation. Thus, (1), (2), (3), (4), (5), (9), (25) and (26), together with the prescribed orientation of the applied magnetic field and the geometry of the flow vessel, complete the specification of the problem for the $O(c)$ modification of the velocity field.

In order to determine the nature of the resultant flow field for arbitrary uni-directional, fully developed, undisturbed flows $(\mathbf{u}^{(0)}, p_0)$ and for arbitrary orientations of the applied magnetic field **H**, we first evaluate the appropriate terms of $\sigma_{ij}'^{(0)}$ for the particular choice $\theta = \frac{1}{2}\pi$ and $\phi = \alpha$. To maintain sufficient generality in the resultant equations, we leave the dynamical variables $\mathbf{e}^{(0)}$ and $\boldsymbol{\omega}^{(0)}$, which enter into the equations, unspecified, except to note that

$$e_{11}^{(0)} = e_{22}^{(0)} = e_{33}^{(0)} = e_{12}^{(0)} = e_{21}^{(0)} = \omega_3^{(0)} = 0$$

for fully developed uni-directional flows (i.e. for $\mathbf{u}^{(0)} = u_3^{(0)}(x_1, x_2)\,\mathbf{i}_3$). When the uni-directional undisturbed velocity profile $(u_3^{(0)})$ is symmetric in x_1 and x_2 as, for example, in the flow through vessels with circular symmetry, the orientation of the magnetic field **H** can be completely specified by the single angle α, which appears explicitly in the equations. In order to recover a particular orientation of the magnetic field in the non-symmetric case, we simply imagine the flow vessel to be rotated about the x_3 axis, hence effectively changing the function $u_3^{(0)}(x_1, x_2)$ defining the undisturbed velocity profile. Therefore, by maintaining the angle α in the analysis, and by carrying the 'non-zero' components of $\mathbf{e}^{(0)}$ and $\boldsymbol{\omega}^{(0)}$ through in symbolic form, we achieve a completely general representation with a minimum of algebraic complexity. Thus, setting $\theta = \frac{1}{2}\pi$ and $\phi = \alpha$, and combining (2)–(5), (25) and (26), we can re-express $(9a, b)$ as

$$\nabla^4\psi = \left[\frac{\partial^2\sigma_{21}'^{(0)}}{\partial x_1^2} + \frac{\partial^2\sigma_{22}'^{(0)}}{\partial x_2\,\partial x_1} - \frac{\partial^2\sigma_{11}'^{(0)}}{\partial x_2\,\partial x_1} - \frac{\partial^2\sigma_{12}'^{(0)}}{\partial x_2^2}\right], \tag{27a}$$

and
$$\mu\nabla^2 w_1 = \frac{\partial p_1}{\partial x_3} + \rho\left(\frac{\partial\psi}{\partial x_2}\frac{\partial w_0}{\partial x_1} - \frac{\partial\psi}{\partial x_1}\frac{\partial w_0}{\partial x_2}\right) - \left(\frac{\partial\sigma_{31}'^{(0)}}{\partial x_1} + \frac{\partial\sigma_{32}'^{(0)}}{\partial x_2}\right), \tag{27b}$$

where

$$\left[\frac{\partial^2\sigma_{21}'^{(0)}}{\partial x_1^2} + \frac{\partial^2\sigma_{22}'^{(0)}}{\partial x_2\,\partial x_1} - \frac{\partial^2\sigma_{11}'^{(0)}}{\partial x_2\,\partial x_1} - \frac{\partial^2\sigma_{12}'^{(0)}}{\partial x_2^2}\right] \equiv A\frac{\partial}{\partial x_1}\left(\frac{\partial\omega_1^{(0)}}{\partial x_2} + \frac{\partial\omega_2^{(0)}}{\partial x_1}\right) + B\frac{\partial}{\partial x_1}\left(\frac{\partial^2 w_0}{\partial x_2^2}\right), \tag{28a}$$

$$\left[\frac{\partial\sigma_{31}'^{(0)}}{\partial x_1} + \frac{\partial\sigma_{32}'^{(0)}}{\partial x_2}\right] \equiv D\frac{\partial^2 w_0}{\partial x_1^2} + E\frac{\partial^2 w_0}{\partial x_2^2}. \tag{28b}$$

Here, the $O(1)$ and $O(c)$ components $u_3^{(0)}$ and $u_3^{(1)}$ have been denoted by w_0 and w_1, respectively, and

$$A \equiv \mu\left[\frac{I_1 - I_2}{I_1 I_2} + \frac{b^2(b^2-a^2)}{(a^2+b^2)^2 J_2 + 2a^2b^2 I_2}\right]\sin 2\alpha,$$

$$B \equiv \mu\left[\frac{2}{I_1}\left\{\frac{5}{3}-\frac{2J_1}{J_2}-\frac{I_1}{I_2}+\left(\frac{2I_1}{I_2}-1-\frac{J_1}{J_2}\right)\cos^2\alpha\right\}\right.$$
$$\left.+\left(\frac{(b^2-a^2)(a^2+b^2)}{(a^2+b^2)^2J_2+2a^2b^2I_2}\right)\left(\frac{a^2-b^2}{a^2+b^2}\cos 2\alpha-1\right)\right]\sin 2\alpha,$$

$$D \equiv 2\mu\left[\frac{\cos^2\alpha}{I_1}+\frac{\sin^2\alpha}{I_2}+\frac{2b^4\sin^2\alpha}{(a^2+b^2)^2J_2+2a^2b^2I_2}\right],$$

$$E \equiv \mu\left[\frac{1}{I_1}\left(\frac{J_1}{J_2}+1\right)\sin^2 2\alpha+\frac{2\cos^2 2\alpha}{I_2}+\left(1+\frac{a^2-b^2}{a^2+b^2}\cos 2\alpha\right)^2\left(\frac{(a^2+b^2)^2}{(a^2+b^2)^2J_2+2a^2b^2I_2}\right)\right].$$

The condition (11 *a*) that there be no secondary flow becomes

$$\left[\frac{\partial^2\sigma_{21}'^{(0)}}{\partial x_1^2}+\frac{\partial^2\sigma_{22}'^{(0)}}{\partial x_1\,\partial x_2}-\frac{\partial^2\sigma_{11}'^{(0)}}{\partial x_1\,\partial x_2}-\frac{\partial^2\sigma_{12}'^{(0)}}{\partial x_2^2}\right] = 0. \tag{29}$$

In the absence of secondary flow, the condition (11*b*), that the resultant unidirectional profile (w_0+cw) be of the same form as for a Newtonian fluid, becomes

$$\left[\frac{\partial\sigma_{31}'^{(0)}}{\partial x_1}+\frac{\partial\sigma_{32}'^{(0)}}{\partial x_2}\right] = \mu f(\alpha,a,b)\,\nabla^2 w_0, \tag{30}$$

where, as before, $\nabla^2 w_0 \equiv$ const. As in the case of spherical particles, the simultaneous satisfaction of these conditions leads directly to an expression,

$$\frac{\mu^*-\mu}{\mu c} = f(\alpha,a,b) = \frac{D(\partial^2 w_0/\partial x_1^2)+E(\partial^2 w_0/\partial x_2^2)}{\mu\nabla^2 w_0}, \tag{31}$$

for the effective viscosity. In the general case in which neither (29) nor (30) is satisfied, the resultant velocity field will exhibit both secondary flow and a modified profile shape for the axial component w_0+cw_1. The nature of the flow, in a particular case, depends on the geometry of the flow vessel (i.e. on $w_0(x_1,x_2)$), on its orientation relative to the applied magnetic field (i.e. on α and the implicit dependence of $w_0(x_1,x_2)$ on the orientation of the vessel about the x_3 axis), and, finally, on the geometry of the particles (i.e. on a/b and hence also on I_1, I_2, J_1, J_2).

6.1. *The existence of secondary flow*

Although we expect secondary motion in the general case, the condition (29) is satisfied in certain exceptional cases which we note here. First of all, when $\alpha = 0$ or $\alpha = \frac{1}{2}\pi$, we have $A = B = 0$, and hence no secondary motion can occur to $O(c)$. These two cases correspond to the applied magnetic field **H** (and hence the axis of revolution of the particle), being perpendicular and parallel, respectively, to the undisturbed velocity vector. When the coefficients A and B are not zero, then the condition (29), together with (28), leads to the conclusion that

$$\frac{\partial}{\partial x_1}\left(\frac{\partial\omega_1^{(0)}}{\partial x_2}+\frac{\partial w_2^{(0)}}{\partial x_1}\right) = 0, \quad \frac{\partial}{\partial x_1}\left(\frac{\partial^2 w_0}{\partial x_2^2}\right) = 0,$$

if secondary motion is not to occur. In particular, the undisturbed vorticity distribution must depend linearly on x_1 and x_2. As noted earlier, this condition is actually satisfied for a fairly wide class of flow situations including the totality of two-dimensional undisturbed motions (i.e. $\mathbf{u}^{(0)} = u_3^{(0)}(x_1 \text{ only})\,\mathbf{i}_3$). In the flow through annuli of arbitrary cross-section, however, the undisturbed vorticity is not linear in x_1 and x_2, and hence, provided the various terms on the right-hand side of (27a) do not cancel, we would expect to observe secondary motion. This is easily confirmed by detailed solutions for the limiting cases where the spheroids either become slender 'rods' ($a/b \to \infty$) or flat 'disks' ($a/b \to 0$).

Before proceeding to a consideration of the nature of the $O(c)$ modification to the undisturbed profile, it is worthwhile to contrast our conclusions for general spheroids with those obtained in § 4 for spherical particles. Thus, we note that in the strong-field limit with *spherical* particles, no secondary could occur for *any* case in which the undisturbed velocity field was uni-directional and fully developed as assumed, whereas for *spheroidal* particles the situation is less clear-cut with the existence or absence of secondary motion being dependent on the geometry of the flow vessel. Since, in all cases,

$$\sigma'_{ij} = \ldots - \tfrac{1}{2}\epsilon_{ijk}\left(\frac{3c}{4\pi ab^2}\right)L_k,$$

it is clear, as stated in the introduction, that the qualitative contribution of the anti-symmetric portion of bulk stress to the bulk velocity field is the same for spheroids as for spheres. By this we mean that, if no secondary motion occurred for spheres, then none would occur for spheroids either, unless caused by the symmetric portions of the bulk stress. Indeed, the non-homogeneous contribution to the right-hand side of (27a) in the circular annulus problem can be shown to arise from the symmetric terms of $\sigma_{ij}'^{(0)}$.

6.2. *The form of the velocity profile for cases of no secondary flow*

In this section we consider the application of (30) to determine the nature of the resulting uni-directional velocity profile for the situations, described in § 6.1, in which there is no secondary motion to $O(c)$. When both (29) and (30) are satisfied, the suspension can be treated, so far as the velocity field is concerned, as a Newtonian fluid with an effective viscosity,

$$\mu^* = \mu(1 + cf(\alpha, a, b)),$$

where $f(\alpha, a, b)$ has been defined in (31). Note that $f(\alpha, a, b)$ depends implicitly on the rotational orientation of the apparatus relative to the x_1, x_2 axes. Provided that the undisturbed motion is fully developed, as we have assumed, we see from (28b) and (30) that the velocity profile will have the same shape as that of a Newtonian fluid, provided that $\partial^2 w_0/\partial x_1^2$ and $\partial^2 w_0/\partial x_2^2$ are individually constant, or equivalently, that the undisturbed vorticity components $\omega_1^{(0)}(\equiv \partial w_0/\partial x_2)$ and $\omega_2^{(0)}(\equiv -\partial w_0/\partial x_1)$ are linear in x_1 and x_2, respectively. This condition is satisfied, with one exception, whenever there is no secondary motion. The exceptional case occurs when $\alpha = 0$, and $\omega_1^{(0)}$ and $\omega_2^{(0)}$ are not linear functions of x_1 and x_2. An example of such a situation is the flow through an annulus of arbitrary cross-section with the magnetic field directed normal to the annulus axis. In this case, no secondary flow occurs, but the resultant uni-directional velocity profile is nevertheless not Newtonian in form. The conclusion that the absence of secondary flow is not a sufficient condition for a Newtonian-like velocity profile is not unlike that obtained for spheres. In both cases, a further condition on the form of the undisturbed vorticity field is required to guarantee a completely Newtonian form for the velocity distribution.

Hence, so far as the qualitative features of the bulk motion are concerned, the primary new contribution, which results from the particles being non-spherical, is that the strong-field approximation is no longer a sufficient condition for the velocity field (to $O(c)$) to remain uni-directional. As pointed out previously, this new feature is clearly a direct contribution of the symmetric terms of the particle stress, since the form of the non-symmetric term is independent of particle shape.

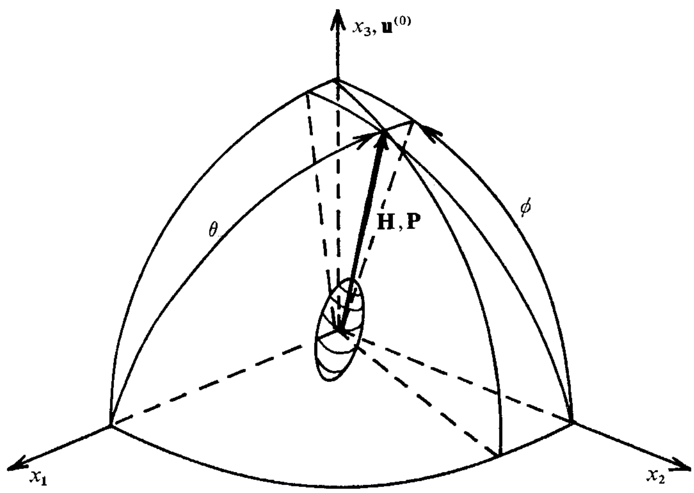

FIGURE 5. The particle orientation relative to the undisturbed velocity vector.

6.3. *The effective viscosity when the velocity field is Newtonian*

Of perhaps greater practical importance than this change in qualitative features is the role of particle shape in establishing the important overall property which governs the volume flow rate for a given imposed pressure gradient. When the velocity profile remains Newtonian in form, this property is simply the effective viscosity of the suspension. A partial investigation of the viscosity of suspensions of rigidly held spheroids is reported by Chaffey & Mason (1965) based on calculations of the extra dissipation induced by the particles. However, this work is limited to a consideration of some special orientations of the particles relative to a simple, plane shear flow (the particle axis of revolution is assumed to be in the plane of the shearing motion). We extend this work to a calculation of the effective viscosity, for arbitrary orientation of the magnetic field (and hence of the particles) and for all vessel geometries in which the velocity field remains completely Newtonian in form. For simplicity, we treat only the limiting cases of slender 'rods' ($a/b \to \infty$) and flat 'disks' ($a/b \to 0$). We thus evaluate the function $f(\alpha, a, b)$ (equation (29)) using the appropriate asymptotic forms of the integral functions I_1, I_2, J_1 and J_2. We recall that, in addition to the explicit dependence of the right-hand side of (31) on a, b and α, the expressions for $\partial^2 w_0/\partial x_1^2$ and $\partial^2 w_0/\partial x_2^2$ will generally change value due to a rotation of the flow vessel about the $\mathbf{u}^{(0)}$ axis.

The exceptions to this implicit dependence occur when the undisturbed velocity profile is circularly symmetric (as in the flow through a circular cylinder), and hence unchanged by such rotations. We denote axes which are fixed in the flow vessel by $\bar{x}_1$ and $\bar{x}_2$. Then the orientation of the vessel can be described in terms of additional angle γ which measures the rotation of the $\bar{x}_2$ axis from the x_2 axis, the latter being fixed in space (cf. figure 6, in which the $\bar{x}_i$ system is shown superposed on the x_i system of figure 5). Hence,

$$u^{(0)} = u_3^{(0)}(\bar{x}_1, \bar{x}_2)\,\mathbf{i}_3 = w_0(\bar{x}_1, \bar{x}_2)\,\mathbf{i}_3,$$

where

$$\left.\begin{aligned} \bar{x}_1 &= x_1 \cos\gamma + x_2 \sin\gamma, \\ \bar{x}_2 &= -x_1 \sin\gamma + x_2 \cos\gamma. \end{aligned}\right\} \tag{32}$$

Of course, an entirely equivalent geometrical description is obtained by considering $\bar{x}_1, \bar{x}_2$ axes to be fixed in space with the orientation of the applied magnetic field varying. Hence, the angles α and γ essentially specify the orientation of the applied field $\mathbf{H}$ relative to the flow vessel which is now considered to be fixed in space. For convenience, we denote

$$\frac{\partial^2 w_0}{\partial \bar{x}_1^2} = G_1, \quad \frac{\partial^2 w_0}{\partial \bar{x}_2^2} = G_2, \quad \frac{\partial^2 w_0}{\partial \bar{x}_1 \partial \bar{x}_2} = G_3 \quad (G_i = \text{const.}), \tag{33}$$

this being consistent with the requirement (30). Combining (28), (31), (32), and (33), we obtain the general expressions for the effective viscosity which we have listed in table 2. These expressions, derived in the respective limits $a/b \rightarrow \infty$ and $a/b \rightarrow 0$, are valid for arbitrary orientations of the applied magnetic field and general uni-directional undisturbed flows, provided that the resultant velocity field remains Newtonian in form (hence, that conditions (29) and (30) are satisfied).

We note that the expressions for μ^* simplify for two-dimensional undisturbed flows, and also for the class of problems in which the undisturbed profile is circularly symmetric. In the first case, the effective viscosity becomes independent of the details of the undisturbed profile, although not of the orientation of the magnetic field relative to $\mathbf{u}^{(0)}$. This follows formally from the fact that G_3, and one of G_2 and G_1, can be made equal to zero (with no loss of generality). If $G_1 = 0$, for example, then γ measures the angle of the field vector $\mathbf{H}$ from the plane of the two-dimensional motion. When the undisturbed profile is circularly symmetric, $G_1 \equiv G_2$ and $G_3 = 0$. Hence, as expected on physical grounds, the dependence of the effective viscosity on γ vanishes. Furthermore, the effective viscosity is independent of the geometry of the flow vessel.

It will be noted that each of the expressions in table 2 is apparently singular for the majority of particle orientations (α, γ). However, it is not entirely clear, at first sight, what interpretation to attach to this result, since, if either of the limits was taken with volume concentration c held fixed, the formal expansion procedure represented by (7) would clearly become invalidated for sufficiently large (or small) values of the axis ratio (a/b). In order for c to be held fixed, however, either the maximum linear dimension of each particle would have to be increased, or else the number density of particles would have to be increased. In either case,

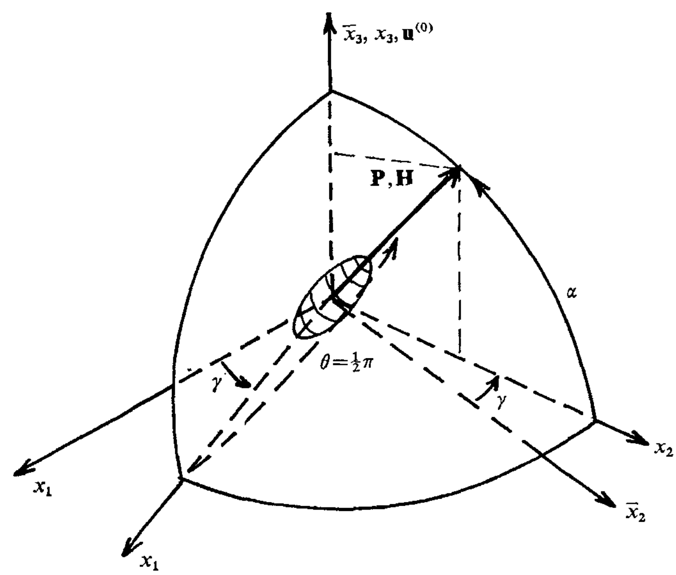

FIGURE 6. The rotation of the co-ordinate system.

Geometry	$(\mu^*-\mu)/(\mu c)$
Slender rods $a/b \gg 1$	$2\left[\frac{\cos^2\gamma G_1+2\sin\gamma\cos\gamma G_3+\sin^2\gamma G_2}{G_1+G_2}\right] + \left[2\cos^2 2\alpha+\frac{3(a/b)^2}{4\log(a/b)}\sin^2 2\alpha+(1+\cos 2\alpha)^2 \times\left(\frac{3a^2}{2b^2\log(a/b)}\right)\right]\times\left[\frac{\sin^2\gamma G_1-2\sin\gamma\cos\gamma G_3+\cos^2\gamma\theta_2}{G_1+G_2}\right]$
Flat disks $a/b \ll 1$	$\left[\frac{8b}{3\pi a}+\left(1+\frac{10b}{7\pi a}\right)\sin^2\alpha\right]\times\left[\frac{\cos^2\gamma G_1+2\sin\gamma\cos\gamma G_3+\sin^2\gamma G_2}{G_1+G_2}\right] + \left[\frac{5b}{3\pi a}\sin^2 2\alpha+\cos^2 2\alpha+(1-\cos 2\alpha)^2\frac{8b}{7\pi a}\right]\times\left[\frac{\sin^2\gamma G_1-2\sin\gamma\cos\gamma G_3+\cos^2\gamma G_2}{G_1+G_2}\right]$

TABLE 2. Effective viscosity

the average interparticle separation in the suspension would decrease so that eventually particle–particle interactions would become important and these have been neglected throughout our investigation. The physical limiting process which is entirely consistant with our theory is, instead, to let $(a/b) \to \infty$ (or $(a/b) \to 0$), with both the maximum dimension a (or b) and the number density of the particles fixed. In these circumstances, volume concentration decreases proportionally to $(a/b)^{-2}$ (or a/b), and the expressions in table 2 remain bounded. Nevertheless, when either (a/b) or (b/a) is large, the results of table 2 suggest that relatively large changes in the effective viscosity may be achieved by changes in the orientation of the magnetic field, and it may be speculated that, in spite of the inherent theoretical difficulties, this effect would be enhanced in magnitude by moderate increases of the particle concentration. This in turn suggests the importance of proper alignment of the external field in minimizing energy comsumption in the transport of such suspensions. Alternatively, it may well be that the variations in effective viscosity could be made large enough to enable the flow rate to be controlled to a useful extent, say by a variable combination of orientation and strength of the applied magnetic field.

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